Large regular QCD coupling at Low Energy ?

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Abstract

The issue is the expediency of the QCD notions use in the low energy region down to the confinement scale, and, in particular, the efficacy of the QCD invariant coupling $\bar{\alpha}_s(Q^2)$ with a minimal analytic modification in this domain. To this goal, we overview a quite recent progress in application of the ghost-free Analytic Perturbative Theory approach (with no adjustable parameters) for QCD in the region below 1 GeV. Among them the Bethe-Salpeter analysis of the meson spectra and spin-dependent (polarization) Bjorken sum rule.
The impression is that there is a chance for the theoretically consistent and numerically correlated description of hadronic events from Z_0 till a few hundred MeV scale by combination of analytic pQCD and some explicit non-perturbative contribution in the spirit of duality.
This is an invitation to the practitioner community for a more courageous use of ghost-free QCD coupling models for data analysis in the low energy region.

1 The pQCD overview

QCD effective coupling $\bar{\alpha}_s$. Common perturbative QCD (pQCD) based upon Feynman diagrams starts with power expansion in $\alpha_s = g_s^2/4\pi \sim 0.1 - 0.4$, the strong interaction parameter analogous to the QED fine structure constant.

In QFT, an important physical notion is an invariant (or effective, or running) coupling function $\bar{\alpha}(Q)$, first mentioned in the QED context by Dirac (1933). In the current practice it was introduced in the basic renormalization group papers of the mid-50s[1].

The one-loop invariant QCD coupling sums up leading order (LO) logs into a geometric progression (with the Bethke[2] convention for the β_k coefficients)

$$\bar{\alpha}_s^{(1)}(Q) = \frac{\alpha_s(\mu)}{1+\alpha_s(\mu)\beta_0\,\ln(\frac{Q^2}{\mu^2})} = \frac{1}{\beta_0\,L}, \quad L = \ln\left(\frac{Q^2}{\Lambda^2}\right)\,; \quad \beta_0 = \frac{33-2\,n_f}{12\pi} > 0\,. \quad (1)$$

At high enough energy (small distance), the QCD interaction diminishes $\bar{\alpha}_s(Q) \sim 1/\ln Q \to 0 \quad \text{as} \quad Q/\Lambda \to \infty; \quad r\,\Lambda \to 0$. This feature is the famous phenomenon of Asymptotic Freedom.

At the same time, eq.(1) obeys unphysical singularity (Landau pole) $\sim 1/(Q^2 - \Lambda^2)$ in the low-energy physical region at $|Q| = \Lambda \sim 400\,\mathrm{MeV}$. Transition to the 2-loop case does not resolve the issue.

The asymptotic freedom behavior $1/\ln Q$ remains dominant in the 2-loop or Next-to-Leading-Order (NLO) case. Here, an explicit expression for $\bar{\alpha}_s$ obtained by iterative approximate solving[1, 3] of differential RG equation can be written down in a compact the "denominator form" (as it was recently motivated in [4])

$$\bar{\alpha}_s^{(2)}(Q) = \frac{1}{\beta_0\, L + \frac{\beta_1}{\beta_0} \ln L}\,; \quad \beta_1(n_f) = \frac{153 - 19 n_f}{24\pi^2} \tag{2}$$

with values $\beta_0(4 \pm 1) = 0.663 \mp 0.053\,;\ \beta_1(4 \pm 1) = 0.325 \mp 0.085\,.$

The QCD scale in the $\overline{\mathrm{MS}}$ scheme $\Lambda^{(n_f)} = \Lambda^{(n_f)}_{\overline{\mathrm{MS}}}$, as obtained from the data happens to be close to the confinement scale $\Lambda^{(4\pm1)} \sim 300 \mp 100$ MeV $\simeq 2\,m_\pi$ or $R_\Lambda \sim 10^{-13}\,\mathrm{cm}$.

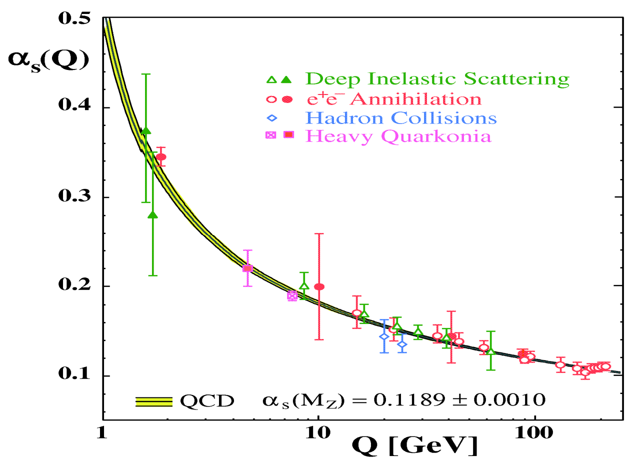

Figure 1: *Effective QCD coupling correlating all the data in the range from a few GeV up to a few hundred GeV. The solid curves correspond to the 2-loop, NLO case. Taken from the Bethke paper [2].*

According to Bethke[2], the 2-loop pQCD approximation (2) turns out to be sufficient for numerical correlation of several dozen of various experiments. Indeed, Fig.1 gives the evidence for *the two-loop pQCD triumph*: the NLO theoretical curve describes quite accurately - within the current experimental and theoretical errors – all the data in the energy range from 5 up to a few hundred GeV.

However, below 5 GeV the correlation is not so persuasive. Moreover, in this region the data on Fig.1 (as well as in the corresponding PDG [5] plot) are rather scanty. The reason is not a shortage of experiments but rather troubles of their theoretical analysis. Among the latter – the issue of unphysical singularities, like the "Landau pole".

As it is well known, the widely-used expressions for effective QCD coupling (like eqs.(1),(2); see also eq.(7) in Ref.[2]) and eq.(9.5) in Ref.[5]) suffer from spurious singularities in the LE physical region at $|Q| \sim \Lambda^{(3)} \sim 400\,\mathrm{MeV}$. This trouble is one of the main embarrassments for the data analysis by pQCD theory below a few GeV.

Unphysical pQCD singularity vs. lattice data. At the same time, numerous lattice simulation results [6] – [8] testify to the regularity of $\bar{\alpha}_s(Q)$ behavior in the region below 1 GeV. Indeed, as it was summarized in papers [9, 10], all the lattice data indicate smooth growth of $\alpha_s$till specific scale $Q = Q_* \sim 400 - 500\,\mathrm{MeV}$ (that is close to $\Lambda^{(3)}$) with typical values $\bar{\alpha}_s(Q_*) \sim 0.5 - 0.8 < 1$.

This means that common iterative solutions of RG eqs., like (1), (2) not only can but should be modified in low-energy region to get rid of singularities and correlate with lattice data.

Modifications of "Common pQCD" in the LE domain. Several attempts to elude the pQCD singularities have been undertaken since the 80s. Among them are the straightforward freezing [11] and a few other, more sophisticated, like glueball mechanism [12] and exponential modification [13]. All of them introduce some model parameters.

Meanwhile, in the mid-90s, an elegant way (free of additional parameters) to resolve this issue was proposed by Solovtsov[1] and collaborators [14] – [16] on the basis of the causality principle implemented in the form of the Källen – Lehmann analyticity for the QCD coupling $\bar{\alpha}_s(Q^2)$. Then, on the ground of Q^2-analyticity, a consistent scheme known as Analytic Perturbation Theory (=APT) has been elaborated [17] – [19] during the last decade.

Below, we give resume of the APT essence (Sect.2) and its application to data (Sect.3) in the above-mentioned troublesome region. These results rise hopes that the Bethke's *issue of two-loop α_s adequacy* can be proliferated to one more order of magnitude – down to a few hundred MeV with the help of analytically modified QCD coupling and some additional nonperturbative means in the spirit of duality.

2 Analytic Perturbation Theory

Here, we start with a sketch of APT. For details see the review papers [20] – [24].

[1]Prof. Igor Solovtsov deceased on July 28, 2007.

2.1 APT - General

As it is well known, the 1st step of improving a renormalized PT result is supplied by the RG Method [1] which allows one to reveal the correct structure of the singularity of a partial solution; in the QFT case – the correct UV and IR asymptotics. Its essence is a technique of restoring the so-called *renormalization*[2] *invariance*. In QFT, the RG-improved results obey a drawback, the unphysical singularity.

In the latter case, the 2nd step, a further improving of RG-invariant PT solution should be used. Its main idea, imposing of the *analyticity imperative* that in turn stems out of the causality condition, was first formulated in the QED context [27]. A more elaborate QCD counterpart, the APT algorithm, is based on the following principles :

- Causality, that results in the analyticity of the effective coupling in the complex Q^2 plane a là Källen-Lehmann representation[3]

  $$\bar{\alpha}_s(Q^2) \to \alpha_{\rm E}(Q^2) = \frac{1}{\pi}\int_0^\infty d\sigma\,\frac{\rho(\sigma)}{\sigma + Q^2 - i\epsilon}\,.$$

  This property provides the absence of spurious singularities.

- Correspondence with perturbative RG-improved input by proper defining $\rho(\sigma) = \mathrm{Im}\,\bar{\alpha}_s(-\sigma)$.

- Representation invariance, i.e., compatibility with linear integral transformations, like a transition from the Euclidean, transfer momentum, picture to the Minkowskian, c.o.m. energy, one:

  $$\alpha_E(Q^2) = Q^2 \int_0^\infty \frac{\alpha_M(s)\,ds}{(s+Q^2)^2}$$

  (or the Fourier transition from $\alpha_E(Q)$ to its Distance image $\alpha_{\rm D}(r)$) that yields [28] non-power functional expansions for observables – see, below eqs.(4),(5).

2.2 The APT Algorithm

Euclidean functions. Euclidean ghost-free expansion functions [28] are defined

$$\mathcal{A}_n(Q^2) = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\,d\sigma, \quad \rho_n(\sigma) = \frac{1}{\pi}\,\mathrm{Im}[\bar{\alpha}_s(-\sigma - i\varepsilon)]^n \qquad (3)$$

[2]Or, more exactly, by the *reparameterization invariance* [25] of a partial solution. Recently, this RG technique has been devised for a class of boundary value problems of classical mathematical physics[26].

[3]For some cases it is implemented in a form of the Jost-Lehmann (see Sec.4 in Ref.[20]) representation.

via powers of $\bar{\alpha}_s$. They form a nonpower set of functions $\{\mathcal{A}_k(Q^2)\}$ that serves as a basis for modified non-power APT expansion of RG invariant objects in the Q picture, like the Adler D-function. The first of these functions can be treated as an Euclidean APT coupling $\alpha_{\mathrm{E}}(Q^2) = \mathcal{A}_1(Q^2)$. In the one-loop case

$$\alpha_{\mathrm{E}}^{(1)}(Q^2) = \frac{1}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right]$$

it differs from the usual one $\alpha_s(Q^2)$ by the term $\sim 1/(\Lambda^2 - Q^2)$ that subtracts the singularity.

Here, higher expansion functions are related by the elegant recurrent relation

$$\mathcal{A}_{n+1}^{(1)}(Q^2) = -\frac{1}{n\beta_0}\frac{d\,\mathcal{A}_n^{(1)}(Q^2)}{d\,\ln Q^2}\,.$$

Minkowskian expansion functions are connected [14, 17, 28] with the Euclidean ones by contour integral and the reverse "Adler transformation"

$$\mathfrak{A}_k(s) = \frac{i}{2\pi}\int_{s-i\varepsilon}^{s+i\varepsilon}\frac{dz}{z}\,\mathcal{A}_k(-z)\,; \quad \mathcal{A}_k(Q^2) = Q^2\int_0^\infty \frac{\mathfrak{A}_k(s)\,ds}{(s+Q^2)^2}\,.$$

The Minkowskian APT coupling $\alpha_M(s) = \mathfrak{A}_1(s)$ in the 1-loop case

$$\alpha_M^{(1)}(s) = \frac{1}{\pi\beta_0}\arccos\left.\frac{L}{\sqrt{L^2+\pi^2}}\right|_{L>0} = \frac{1}{\pi\,\beta_0}\arctan\frac{\pi}{L}\,, \quad L = \ln(s/\Lambda^2)$$

is quantitatively close to the Euclidean APT one; see Fig.2.

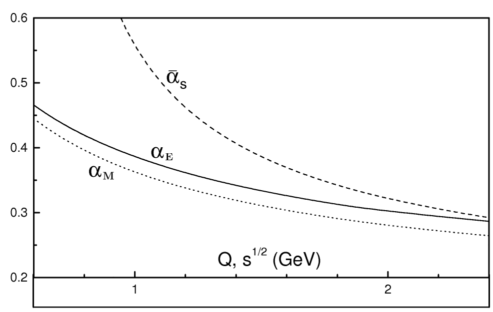

Figure 2: *Comparison of singular $\bar{\alpha}_s$ coupling with Euclidean α_E and Minkowskian α_M in a few GeV region. Taken from paper [18].*

Non-power APT - Loop and RS Stability. In APT, instead of universal power-in-$\bar{\alpha}_s$ ($\bar{\alpha}_s(Q^2)$ or $\bar{\alpha}_s(s)$) series

$$d_{pt}(Q^2/s) = d_1\bar{\alpha}_s(Q^2/s) + d_2\,\bar{\alpha}_s^2 + 0(\bar{\alpha}_s^3)\,,$$

one should use for each representation its own particular non-power expansion

$$d_{\rm an}(Q^2) = d_1\,\alpha_E(Q^2) + d_2\,\mathcal{A}_2(Q^2) + d_3\,\mathcal{A}_3(Q^2) + \ldots\ , \tag{4}$$

$$r_\pi(s) = \ d_1\,\alpha_M(s) \ + \ d_2\,\mathfrak{A}_2(s) \ + \ d_3\,\mathfrak{A}_3(s) \ + \ldots \tag{5}$$

that provides better loop convergence and practical RS independence of observables. The 3rd terms in (3), (4) contribute to observables less than 5 % [21]. Again the 2-loop (NLO) level is practically sufficient.

Fractional APT. In computation of higher-order corrections to inclusive and exclusive processes one deals with non-integer fractional powers of QCD coupling. In such a case, special *fractional* generalization has been devised[29] and successively applied to pion form factor [30] and to the Higgs boson decay into a $b\bar{b}$ pair [31].

2.3 APT functions at LE region

Comparison of APT Euclidean α_E and Minkowskian α_M couplings reveals that below 2-3 GeV scale they, being close to each other, differ seriously from the common singular $\bar{\alpha}_s$ – see Fig.2. Qualitatively, the same is true for higher expansion functions.

The APT RenormScheme- and loop- stability. In Fig.3, we give Euclidean APT coupling in the one-, two- and three-loop (NNLO) approximations taken in the $\overline{\rm MS}$ scheme.

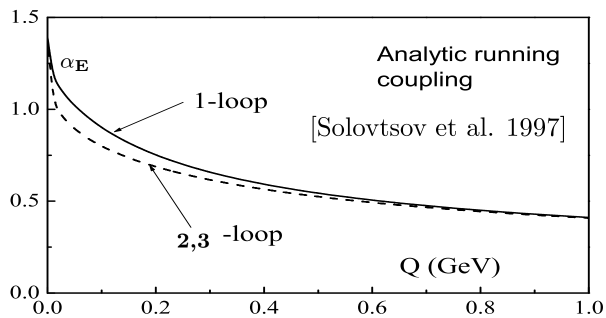

Figure 3: *Euclidean APT coupling α_E in the 1-, 2- and 3-loop cases for the $\overline{\rm MS}$ scheme. Taken from paper [18].*

A beautiful feature of these curves is their relative loop stability. The two-loop curve below 1 GeV differs from the three-loop one by less than 3 per cent. Hence,

the APT two-loop (NLO) curve is accurate enough for practical use at three-flavor region. This correlates with the Bethke's thesis for higher energies.

In a real QCD case, one has to take into account the proper conjunction of regions with different values of effective flavour number n_f . This, a bit subtle issue was elaborated in [19]. From the practical point of view, one should use common matching conditions for recalculation of $\Lambda^{(n_f)}$ values at the quark threshold crossing. Resulting Euclidean functions $\mathcal{A}_k$ turn out to be smooth in the threshold vicinity, while Minkowskian ones $\mathfrak{A}_k$ remaining continuous have jumps in derivatives.

Recall here that all this is valid for simple APT functions *without additional parameters.* This version is known as a *minimal APT*. Below, we shortly mention its massive generalization which contains an additional fitting parameter.

The "massive" APT modification. A quite natural ansatz has been added to the minimal APT formalism in [32]. There, the lower limit in the Kállen-Lehmann integral Eq.(3) was changed from zero to $m^2 > 0$. This parameter reminiscent of pion mass m_π squared can be used for the data fitting (as in Fig.4).

3 Low energy APT application

The APT approach during the decade of it existence has been applied to a number of low energy (above 1-2 GeV) hadronic observables. One has to mention here sum rules [33, 34], $e^+ e^-$ inclusive hadron annihilation [35], τ [36] and Υ decays[37], above-mentioned formfactors[30, 31], and some others. For details one could address to review papers [21, 22]. Below, we shortly overview quite fresh APT applications to processes in a rather low-energy region $\lesssim 1\,\mathrm{GeV}$.

3.1 APT and bosonic spectrum

APT + Bethe-Salpeter formalism. Here, is a summary of recent analysis [38] of the meson spectrum by combination of the Bethe-Salpeter equation for the $(q, \bar{q})$ system with the APT approach.

By use of the 3-dimensional reduction, the BS eq. takes the form of an eigenvalue equation for a squared bound state mass

$$M^2 = M_0^2 + U_{\mathrm{OGE}} + U_{\mathrm{Conf}} \,,$$

with $M_0 = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}$ – kinematic term, U_{Conf} – *confining potential*, U_{OGE} – one-gluon exchange potential $\sim$ QCD coupling

$$\langle \mathbf{k} | U_{\mathrm{OGE}} | \mathbf{k}' \rangle = \alpha_s(\mathbf{Q}^2)\, M_{\mathrm{OGE}}(\mathbf{Q} = \mathbf{k} - \mathbf{k}' \,, \mathbf{k}) \,.$$

For a given bound state a, one has (for details see Refs.[40])

$$m_a^2 = \langle\phi_a|M_0^2|\phi_a\rangle + \langle\phi_a|U_{\rm OGE}|\phi_a\rangle + \langle\phi_a|U_{\rm Conf}|\phi_a\rangle \,.$$

Last two relations allow one to extract $\alpha_s(Q_a^2)$ values for a low enough momentum transfer region $100\,{\rm MeV} < Q_a < 1\,{\rm GeV}$.

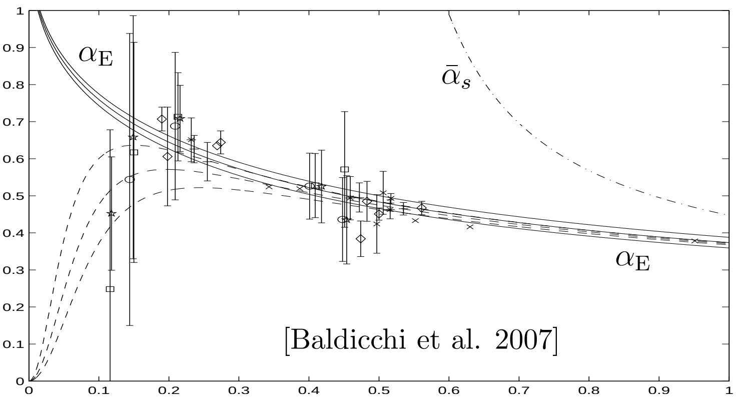

Figure 4: *Comparison of α_s from meson spectrum (points with error bars) and 3-loop $\alpha_{\rm E}$ at $\Lambda^{(3)}_{n_f=3} = (417 \pm 42)$ MeV (3 solid curves). Singular 3-loop $\bar{\alpha}_s$ coupling (dot-dashed) is excluded by data. Dashed lines correspond to the massive APT version [32] with $m \sim 40\,{\rm MeV}$. Taken from paper [38].*

Results of α_s extraction from the bosonic spectrum are given in Fig.4. One sees that meson spectrum data roughly follow a bunch of three α_E curves for $\Lambda^{(3)}_{n_f=3} = (417 \pm 42)\,{\rm MeV}$ corresponding to the 2006 world average $\bar{\alpha}_s(M_Z^2) = 0.1189 \pm 0.0010$. There is also a slight hint at the tendency for BS-extracted $\alpha_s$values at $Q < 200\,{\rm MeV}$ to diminish in the IR limit. The dashed curves on Fig.4 just relate to this possibility. However, in our opinion, such a scenario is supported only by data from the D and F orbital excitations of the $(q, \bar{q})$ system. They have big error bars and some of them are subject to a doubtful interpretation. If we exclude higher states and limit ourselves to the S and P ones, the resulting picture will change.

APT vs S and P data. In Fig.5, we show the picture without higher orbital D and F excitations. This limited set of data with small error bars quite nicely follows the minimal APT coupling curve with the world average $\Lambda^{(3)}_{n_f=3} = 417$ MeV value.

3.2 Bjorken sum rule

Fresh 2006 Jefferson Lab data on the Bjorken Sum Rule for the moment of the spin-dependent structure function $\Gamma_{p-n,1}$ at $0.1 < Q^2 < 3\,{\rm GeV}^2$ were analysed recently

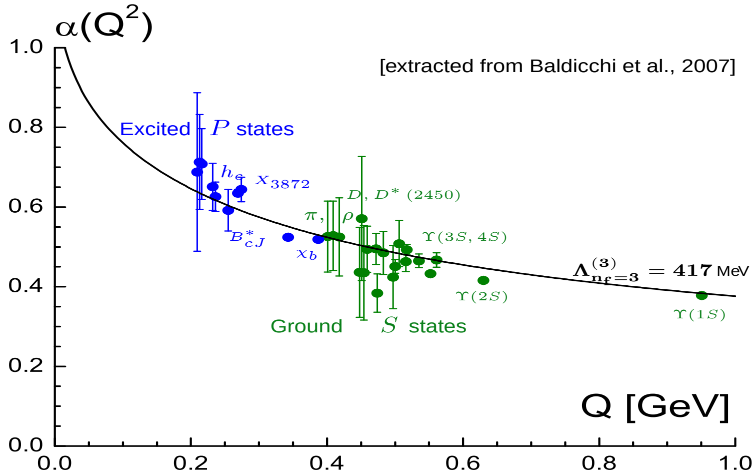

Figure 5: The APT $\alpha_{\rm E}(Q^2)$ coupling correlated with the world average vs. α_s^{exp} from the S,P states of the $(q,\bar{q})$ system. Evidence for evolution below 500 MeV.

in the NLO approximation[41]. Higher twist (HT) values extracted within the APT provide evidence for better convergence of HT series as compared to the standard pQCD. As a final result, a reasonable quantitative description of the data down to 350 MeV was achieved.

Together with the meson spectra evidence Fig.5, this produces an impression that minimal APT allows one to enlarge the domain of analytic perturbative QCD (supported by transparent non-perturbative elements) description of hadronic events down to a few hundred MeV.

4 APT in QCD: Conclusion

Meson spectrum data analyzed by the Milano BS-technique with the one-gluon exchange potential and confinement ansatz result[39] in a possibility to extract the QCD coupling $\bar{\alpha}_s(Q)$ values in the LE domain of momentum transfer $Q < 1\,$GeV. In a recent research it was shown [38, 40] that the use of ghost-free analytic QCD Euclidean coupling α_E in this analysis yields rather an intriguing correlation (shown in Figs.4 and 5) of the "meson spectrum α_s values" in the region $250\,\text{MeV} \lesssim Q < 1\,\text{GeV}$ with the world average $\bar{\alpha}_s(M_Z^2)$.

Along with this, the arena for the APT nonpower expansion results for the Bjorken sum rule[41] is also ranging down as far as to the $\sim 300 - 400\,$MeV scale.

Both the results –

- exclude common α_s singular behavior and smooth "freezing" below 1 GeV,
- support minimal APT extension of pQCD, giving hope for a quasi-perturbative consistent quantitative picture from 200 GeV to 200-300 MeV.

Due to this, there appears a chance for the real possibility of consistent theoretical analysis of hadronic processes in the low-energy region, the chance that is based on two elements:
– the procedure of getting rid of spurious singularities, by some low-energy modification of pQGD, like the APT one;
– addition of some appropriate nonperturbative elements in the spirit of parton-hadron duality, like confinement ansatz and higher twist contribution.

We appeal to the QCD practicing community for a more regular use of ghost-free QCD coupling models for data analysis in the low-energy region below 1 –2 GeV. Just in this region theoretical errors quite often exceed the experimental ones.

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